MathAA_21_SL_Summer_2021_Q1

Solution

1. Indefinite Integration

To find the **indefinite integral** of the linear function $6x + 7$, we apply the **linearity property** of integration and the **power rule** for integration, which states that $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ for $n \neq -1$.

$$\begin{aligned}\int (6x+7)dx &= \int 6x\,dx + \int 7\,dx \\ &= 6\int x^1\,dx + 7\int 1\,dx \\ &= 6\left(\frac{x^{1+1}}{1+1}\right) + 7x + C \\ &= 6\left(\frac{x^2}{2}\right) + 7x + C \\ &= 3x^2 + 7x + C\end{aligned}$$

where C is the **constant of integration**.

$$\boxed{\int (6x+7)dx = 3x^2 + 7x + C}$$

2. Finding the Specific Function

Given the derivative $f'(x) = 6x + 7$, the general form of the function $f(x)$ is the **antiderivative** found in part (a):

$$f(x) = 3x^2 + 7x + C$$

To find the specific value of C, we use the provided **initial condition** $f(1.2) = 7.32$.

- Substitute $x = 1.2$ into the expression for $f(x)$:

$$\begin{aligned}f(1.2) &= 3(1.2)^2 + 7(1.2) + C \\ 7.32 &= 3(1.44) + 8.4 + C \\ 7.32 &= 4.32 + 8.4 + C \\ 7.32 &= 12.72 + C\end{aligned}$$

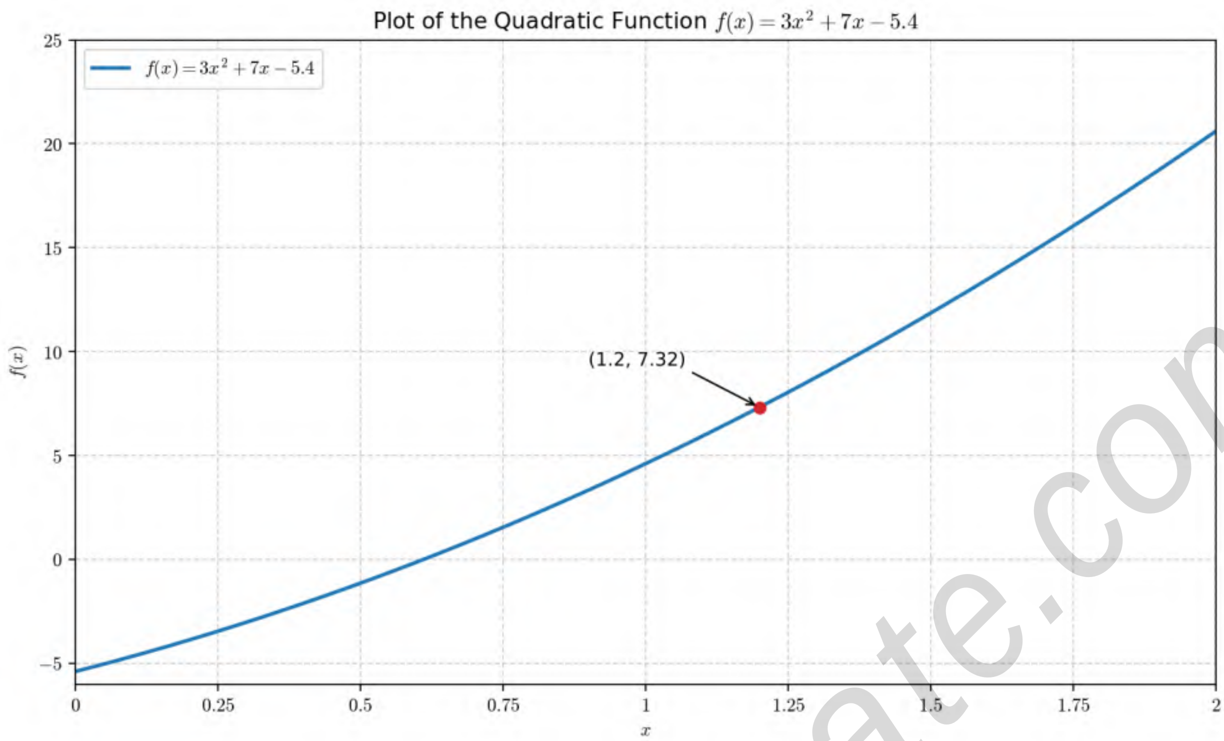
- Solve for C:

$$\begin{aligned}C &= 7.32 - 12.72 \\ C &= -5.4\end{aligned}$$

- Substitute C back into the general equation to obtain the final function:

$$f(x) = 3x^2 + 7x - 5.4$$

www.exam-mate.com



$$\boxed{f(x) = 3x^2 + 7x - 5.4}$$

www.exam-mate.com

MathAA_21_SL_Summer_2021_Q2

Solution

To analyze the relationship between the variables x and y, we utilize the method of **linear regression** to determine the line of best fit.

1. Calculation of Regression Parameters The regression line is given by the equation $y = ax + b$. Using the provided data points:
$(3.3, 6.3), (6.9, 8.1), (11.9, 8.4), (13.4, 11.6), (17.8, 10.3), (19.6, 12.9), (21.8, 13.1), (25.3, 17.3)$

- The slope a is calculated using the formula $a = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - (\sum x)^2}$.
- The y-intercept b is calculated using $b = \bar{y} - a\bar{x}$.

From the statistical analysis of the 8 data points:

$$a \approx 0.43775...$$
$$b \approx 4.5028...$$

Rounding to three significant figures:

$$a = 0.438$$
$$b = 4.50$$

2. Prediction for $x = 18$ Using the regression model $y = 0.43775x + 4.5028$:

$$\begin{aligned} y &= 0.43775(18) + 4.5028 \\ &= 7.8795 + 4.5028 \\ &= 12.3823 \end{aligned}$$

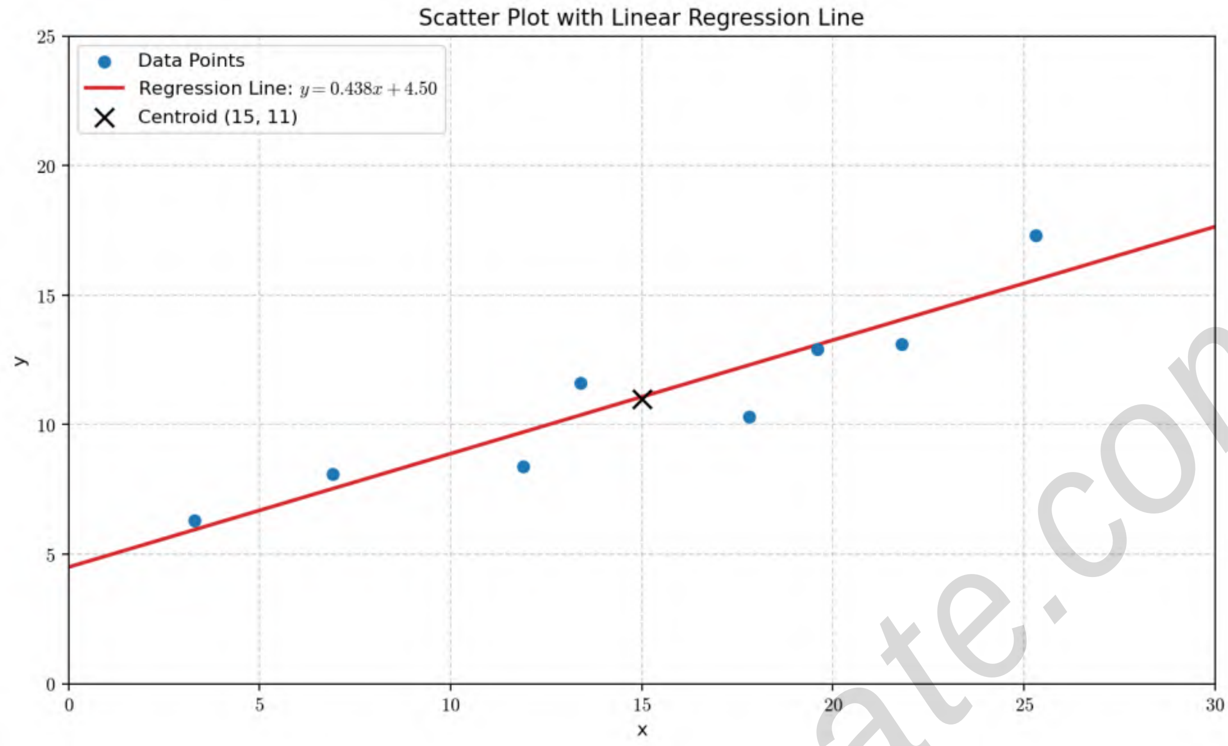
Rounding to three significant figures, $y = 12.4$.

3. Mean Values The **centroid** of the data $(\bar{x}, \bar{y})$ is a point through which the regression line must pass.

$$\bar{x} = \frac{3.3 + 6.9 + 11.9 + 13.4 + 17.8 + 19.6 + 21.8 + 25.3}{8} = 15.0$$
$$\bar{y} = \frac{6.3 + 8.1 + 8.4 + 11.6 + 10.3 + 12.9 + 13.1 + 17.3}{8} = 11.0$$

4. Line of Best Fit To draw the **line of best fit**, we plot the centroid $(15, 11)$ and the y-intercept $(0, 4.50)$, or another predicted point such as $(25, 15.45)$, and draw a straight line through them.

www.exam-mate.com



(a) $a = 0.438, b = 4.50$
(b) $y = 12.4$
(c) $\bar{x} = 15.0, \bar{y} = 11.0$
(d) Line drawn through $(15, 11)$ and $(0, 4.50)$

www.exam-mate.com

MathAA_21_SL_Summer_2021_Q3

Solution

Let the mass of a bag of sugar be represented by the random variable X. According to the problem, X follows a **normal distribution** with a mean $\mu = 1000\,\mathrm{g}$ and a standard deviation $\sigma = 3.5\,\mathrm{g}$. We denote this as $X \sim N(1000, 3.5^2)$.

1. Probability that a bag is rejected A bag is rejected if its mass X is less than $995\,\mathrm{g}$. To find this probability, we convert the value to a **z-score** using the formula $Z = \frac{X-\mu}{\sigma}$.

- Calculate the z-score for $X = 995$:

$$\begin{aligned} z &= \frac{995 - 1000}{3.5} \\ &= \frac{-5}{3.5} \\ &\approx -1.42857 \end{aligned}$$

- The probability $P(X < 995)$ is equivalent to $P(Z < -1.42857)$. Using the standard normal distribution table or a calculator:

$$\begin{aligned} P(X < 995) &= \Phi(-1.42857) \\ &\approx 0.07656 \end{aligned}$$

  Rounding to three significant figures, the probability is 0.0766.

$\boxed{0.0766}$

2. Estimated number of rejected bags in a sample of 100 The number of rejected bags in a random sample follows a **binomial distribution** $B(n, p)$, where $n = 100$ and $p \approx 0.07656$. The expected value (estimate) is given by $E = n \cdot p$.

- Calculation:

$$\begin{aligned} E &= 100 \times 0.07656 \\ &= 7.656 \end{aligned}$$

  Rounding to the nearest whole number for an estimate of items:

$\boxed{8}$

3. Conditional probability for non-rejected bags We need to find the probability that a bag has a mass greater than $1005\,\mathrm{g}$, given that it is not rejected ($X \geq 995$). This is a **conditional probability** problem defined by:

$$P(X > 1005 \mid X \geq 995) = \frac{P(X > 1005 \cap X \geq 995)}{P(X \geq 995)}$$

- Since the condition $X > 1005$ is a subset of $X \geq 995$, the numerator simplifies to $P(X > 1005)$.
- Calculate $P(X > 1005)$: The z-score for 1005 is $z = \frac{1005-1000}{3.5} = \frac{5}{3.5} \approx 1.42857$. Due to the symmetry of the normal distribution, $P(X > 1005) = P(X < 995) \approx 0.07656$.

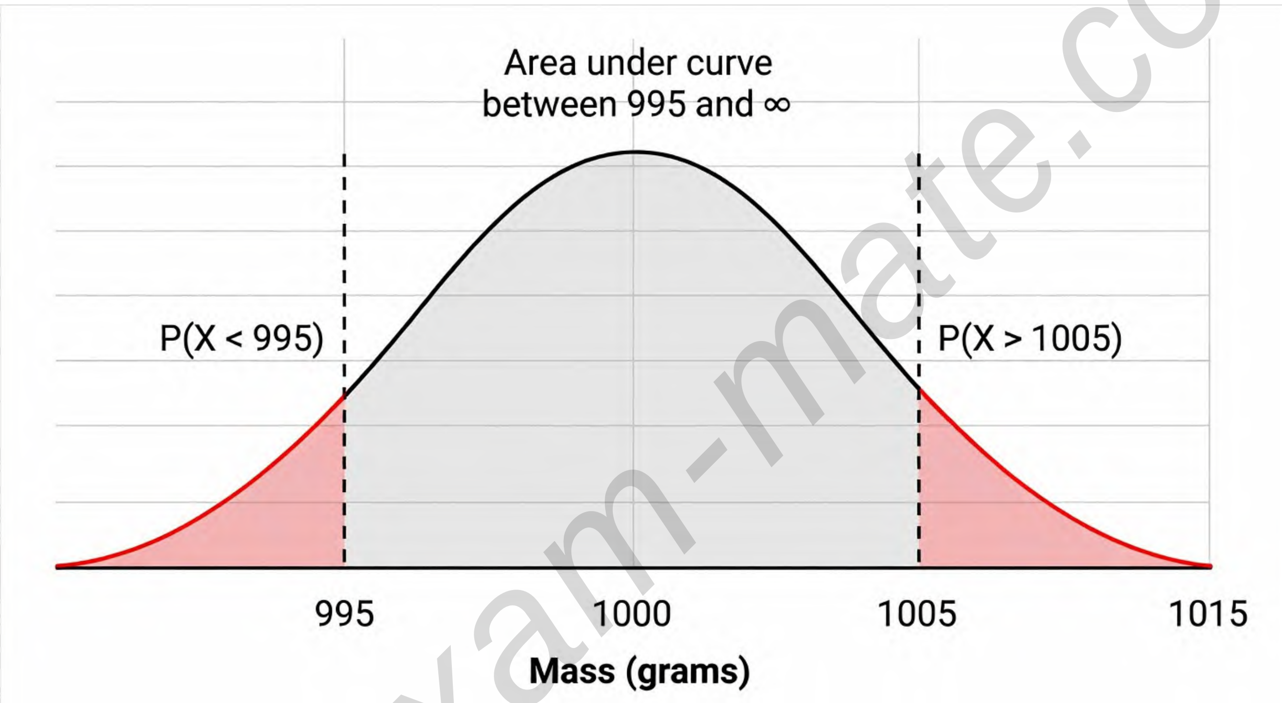
- Calculate $P(X \geq 995)$:

$$\begin{aligned} P(X \geq 995) &= 1 - P(X < 995) \\ &= 1 - 0.07656 \\ &= 0.92344 \end{aligned}$$

- Calculate the conditional probability:

$$\begin{aligned} P(X > 1005 \mid X \geq 995) &= \frac{0.07656}{0.92344} \\ &\approx 0.08291 \end{aligned}$$

Rounding to three significant figures, the probability is 0.0829.



$\boxed{0.0829}$

www.exam-mate.com

MathAA_21_SL_Summer_2021_Q4

Solution

To model the height $h(t)$ of point P on the Ferris wheel using a **cosine function**, we analyze the geometric properties and the motion parameters provided.

1. Identification of Geometric Parameters The Ferris wheel has a diameter $D = 110\,\mathrm{m}$ and its lowest point is $H_{\min} = 10\,\mathrm{m}$ above the ground.

- The radius r of the wheel is:

$$r = \frac{D}{2} = \frac{110}{2} = 55\,\mathrm{m}$$

- The maximum height $H_{\max}$ reached by point P is:

$$H_{\max} = H_{\min} + D = 10 + 110 = 120\,\mathrm{m}$$

- The **amplitude** a of the oscillation corresponds to the radius. However, since the point P starts at the lowest point ($t = 0$), the cosine function (which normally starts at a maximum) must be reflected. Thus:

$$\mid a \mid = r = 55$$

  Since $h(0) = a\cos(0) + c = a + c$ must equal the minimum height, and c is the midline, a must be negative to start at the minimum.

$$a = -55$$

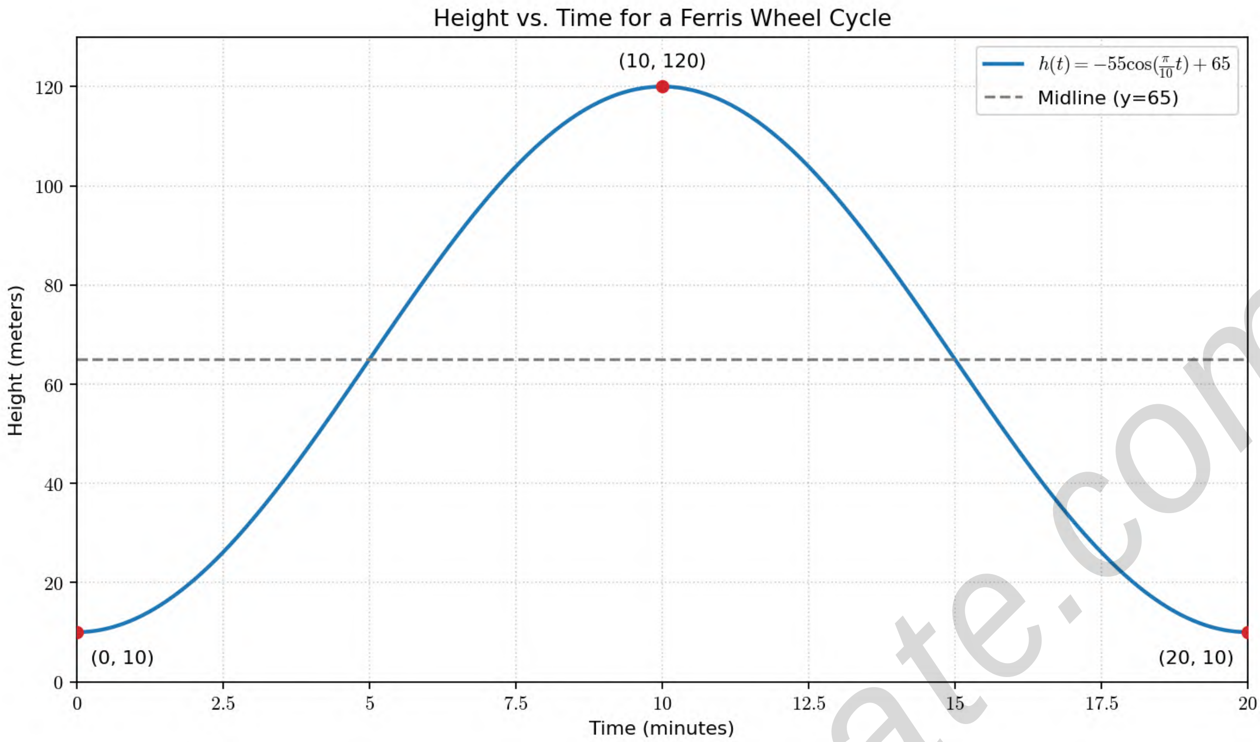
2. Determination of the Vertical Shift (Midline) The **vertical shift** c represents the height of the center of the Ferris wheel above the ground.

$$\begin{aligned} c &= \frac{H_{\max} + H_{\min}}{2} \\ &= \frac{120 + 10}{2} \\ &= 65\,\mathrm{m} \end{aligned}$$

3. Determination of the Angular Frequency The wheel completes one revolution in $T = 20\,\mathrm{minutes}$. The parameter b relates to the **period** T of the trigonometric function.

$$\begin{aligned} b &= \frac{2\pi}{T} \\ &= \frac{2\pi}{20} \\ &= \frac{\pi}{10} \end{aligned}$$

www.exam-mate.com



4. Final Model Assembly Substituting the calculated values into the general form $h(t) = a\cos(bt) + c$:

- $a = -55$
- $b = \frac{\pi}{10}$
- $c = 65$

Verifying at $t = 0$: $h(0) = -55\cos(0) + 65 = -55(1) + 65 = 10\,\text{m}$, which matches the lowest point.

$$\boxed{a = -55, b = \frac{\pi}{10}, c = 65}$$

www.exam-mate.com

MathAA_21_SL_Summer_2021_Q5

Solution

The problem involves analyzing the **kinematics** of a particle moving in a straight line. The velocity function is given by $v(t) = t\sin t - 3$ for the interval $0 \le t \le 10$.

1. Finding the smallest value of t for which the particle is at rest

- A particle is at rest when its **instantaneous velocity** is zero. We solve for t in the equation:

$$v(t) = t\sin t - 3 = 0$$

- Using numerical methods (such as a graphing calculator or **Newton's method**) to find the roots of $t\sin t - 3 = 0$ on the interval $[0, 10]$:
  - The first root occurs between $t = 6$ and $t = 7$ as seen on the graph.
  - Solving numerically: $t \approx 6.744167$.
- Rounding to three significant figures:

$$\boxed{t = 6.74\,\text{s}}$$

2. Finding the total distance travelled by the particle

- The **total distance** d travelled by a particle over a time interval $[a, b]$ is the integral of the **speed** (the magnitude of velocity):

$$d = \int_0^{10} |\ v(t)\ |\,dt = \int_0^{10} |\ t\sin t - 3\ |\,dt$$

- We evaluate this integral numerically:

$$\begin{aligned} d &= \int_0^{6.744} (3 - t\sin t)\,dt + \int_{6.744}^{9.088} (t\sin t - 3)\,dt + \int_{9.088}^{10} (3 - t\sin t)\,dt \\ &\approx 37.09685 \end{aligned}$$

- Rounding to three significant figures:

$$\boxed{d = 37.1\,\text{m}}$$

3. Finding the acceleration of the particle when $t = 7$

- **Acceleration** $a(t)$ is the first derivative of velocity with respect to time:

$$a(t) = \frac{dv}{dt}$$

- Applying the **product rule** to $v(t) = t\sin t - 3$:

$$\begin{aligned} a(t) &= \frac{d}{dt}(t\sin t) - \frac{d}{dt}(3) \\ &= (1 \cdot \sin t + t \cdot \cos t) - 0 \\ &= \sin t + t\cos t \end{aligned}$$

- Evaluating at $t = 7$:

$$\begin{aligned} a(7) &= \sin(7) + 7\cos(7) \\ &\approx 0.656986 + 7(0.753902) \\ &\approx 0.656986 + 5.277315 \\ &\approx 5.934301 \end{aligned}$$

- Rounding to three significant figures:

$$\boxed{a(7) = 5.93\,\mathrm{m}\cdot\mathrm{s}^{-2}}$$

www.exam-mate.com

MathAA_21_SL_Summer_2021_Q6

Solution

To find the value of n in the expansion of $(3+x^2)^{n+1}$, we utilize the **Binomial Theorem**.

1. General Term of the Expansion The general term T_{r+1} in the expansion of $(a+b)^N$ is given by:

$$T_{r+1} = \binom{N}{r} a^{N-r} b^r$$

In this problem, we have:

- $a = 3$
- $b = x^2$
- $N = n+1$

Substituting these values, the general term is:

$$T_{r+1} = \binom{n+1}{r} (3)^{(n+1)-r} (x^2)^r = \binom{n+1}{r} 3^{n+1-r} x^{2r}$$

2. Identifying the Coefficient of x^4 We are interested in the term containing x^4. Setting the power of x equal to 4:

$$2r = 4$$
$$r = 2$$

The coefficient of x^4 is obtained by substituting $r=2$ into the general term expression:

$$\text{Coefficient} = \binom{n+1}{2} 3^{(n+1)-2} = \binom{n+1}{2} 3^{n-1}$$

3. Solving for n We are given that the coefficient is 20412. Thus:

$$\binom{n+1}{2} 3^{n-1} = 20412$$

Expanding the **binomial coefficient** $\binom{n+1}{2} = \frac{(n+1)n}{2}$:

$$\frac{(n+1)n}{2} \cdot 3^{n-1} = 20412$$
$$(n+1)n \cdot 3^{n-1} = 40824$$

To solve for $n \in \mathbb{Z}^+$, we test integer values for n:

- If $n = 6$: $(7)(6) \cdot 3^5 = 42 \cdot 243 = 10206$ (Too low)
- If $n = 7$: $(8)(7) \cdot 3^6 = 56 \cdot 729 = 40824$ (Correct)

Verification via calculation:

www.exam-mate.com

$$\begin{aligned}(7+1)(7)\cdot 3^{7-1} &= 8\cdot 7\cdot 3^6\\ &= 56\cdot 729\\ &= 40824\end{aligned}$$

The equation holds true for $n = 7$.

$\boxed{n = 7}$

www.exam-mate.com

MathAA_21_SL_Summer_2021_Q7

Solution

1. Amelia’s Investment Analysis

Amelia’s investment follows the **compound interest** formula for annual compounding:

$$A = P(1+r)^n$$

where $P = 9000$, $r = 0.07$, and n is the number of years.

- **(i) Value after 5 years** Substituting the values into the formula:

$$\begin{aligned} A &= 9000(1+0.07)^5 \\ &= 9000(1.07)^5 \\ &\approx 12623.00 \end{aligned}$$

  Rounding to the nearest hundred dollars: $\boxed{\$12600}$

- **(ii) Years to reach the target** We set the target amount $A = 20000$ and solve for n:

$$\begin{aligned} 20000 &= 9000(1.07)^n \\ \frac{20}{9} &= (1.07)^n \\ \ln\left(\frac{20}{9}\right) &= n\ln(1.07) \\ n &= \frac{\ln(20/9)}{\ln(1.07)} \\ &\approx 11.82 \end{aligned}$$

  Since interest is compounded annually, we round up to the next full year. $\boxed{12 \text{ years}}$

2. Bill’s Investment Analysis

Bill’s account uses **monthly compounding**;. The formula is:

$$A = P\left(1+\frac{r}{100\cdot 12}\right)^{12n}$$

Given $P = 9000$, $A = 20000$, and $n = 10$:

www.exam-mate.com

$$20000 = 9000\left(1 + \frac{r}{1200}\right)^{120}$$
$$\frac{20}{9} = \left(1 + \frac{r}{1200}\right)^{120}$$
$$\left(\frac{20}{9}\right)^{1/120} = 1 + \frac{r}{1200}$$
$$\frac{r}{1200} = \left(\frac{20}{9}\right)^{1/120} - 1$$
$$r = 1200\left[\left(\frac{20}{9}\right)^{1/120} - 1\right]$$
$$\approx 8.0119$$

Rounding to two decimal places to ensure the target is reached: $\boxed{r = 8.02\%}$

3. Chris's Savings Strategy

Chris's savings form a **geometric series** where the first term a is the initial deposit and the common ratio is $r = 0.5$.

- **(i) Proof of target impossibility** The total amount saved over an infinite period is the sum to infinity:

$$S_\infty = \frac{a}{1-r}$$

With $a = 9000$ and $r = 0.5$:

$$S_\infty = \frac{9000}{1-0.5} = \frac{9000}{0.5} = 18000$$

Since the maximum possible sum $18000 < 20000$, Chris will never reach the target.

- **(ii) Initial deposit for 5-year target** The sum of a geometric series for n terms is $S_n = \frac{a(1-r^n)}{1-r}$. We set $S_5 = 20000$, $r = 0.5$, and $n = 5$:

$$20000 = \frac{a(1-0.5^5)}{1-0.5}$$
$$20000 = \frac{a(1-0.03125)}{0.5}$$
$$20000 = \frac{a(0.96875)}{0.5}$$
$$20000 = 1.9375a$$
$$a = \frac{20000}{1.9375}$$
$$\approx 10322.58$$

Rounding to the nearest dollar: $\boxed{\$10323}$

www.exam-mate.com

MathAA_21_SL_Summer_2021_Q8

Solution

Based on the geometry provided in the diagrams and the problem description, we proceed as follows:

1. Expression for r in terms of θ The length of an arc in a circle is given by the formula $s = r\theta$, where θ is in radians.

- Given the arc length $DE = 28\,\text{m}$ and the radius of the sector is r:

$$28 = r\theta$$
$$r = \frac{28}{\theta}$$

$$\boxed{r = \frac{28}{\theta}}$$

2. Area of the field reachable by the horse The area reachable by the horse consists of two parts: the circular sector PDE and the triangle PAE.

- The area of the **circular sector** PDE is:

$$A_{\text{sector}} = \frac{1}{2}r^2\theta$$

- In triangle PAE, we have sides $PA = r$ and $PE = r$. The angle $\angle APE$ is the supplement to θ along the straight fence line, so $\angle APE = \pi - \theta$. The area of the triangle is:

$$A_{\text{triangle}} = \frac{1}{2}(PA)(PE)\sin(\pi - \theta) = \frac{1}{2}r^2\sin\theta$$

- Total Area A:

$$A = \frac{1}{2}r^2\theta + \frac{1}{2}r^2\sin\theta$$
$$= \frac{1}{2}r^2(\theta + \sin\theta)$$

- Substitute $r = \frac{28}{\theta}$:

$$A = \frac{1}{2}\left(\frac{28}{\theta}\right)^2(\theta + \sin\theta)$$
$$= \frac{1}{2}\left(\frac{784}{\theta^2}\right)(\theta + \sin\theta)$$
$$= \frac{392}{\theta^2}(\theta + \sin\theta)$$

3. Finding the value of θ Given the area $A = 460\,\text{m}^2$:

$$460 = \frac{392}{\theta^2}(\theta + \sin\theta)$$

Using numerical methods to solve for θ in the interval $(0, \pi)$: $\boxed{\theta \approx 1.44\,\text{rad}}$

www.exam-mate.com

4. Size of $D\widehat{A}E$ In the isosceles triangle PAE, the angle at P is $\phi = \pi - \theta$. Since $PA = PE = r$, the base angles $\angle PAE$ and $\angle PEA$ are equal:

$$\begin{aligned}\angle PAE &= \frac{\pi - (\pi - \theta)}{2} \\ &= \frac{\theta}{2}\end{aligned}$$

Using $\theta \approx 1.4391$: $\boxed{D\hat{A}E \approx 0.720\,\mathrm{rad}}$

5. Geometry of the new fence

- **(i) Find the size of $A\widehat{B}C$** The **bearing** of B from A is 195°. This means the angle from the North line at A clockwise to line AB is 195°. The interior angle at A relative to the South line is $195^\circ - 180^\circ = 15^\circ$. Since C is due West of B, the line CB is horizontal. Let the North-South line through B be L. The angle between AB and L is 15° (alternate interior angles). Since CB is perpendicular to L:

$$\begin{aligned}A\hat{B}C &= 90^\circ + 15^\circ \\ &= 105^\circ\end{aligned}$$

$\boxed{A\hat{B}C = 105^\circ}$

- **(ii) Length of the new fence BC** We use the **Sine Rule** in triangle ABC. We know $AC = 800\,\mathrm{m}$ and $\angle ABC = 105^\circ$. We need $\angle BAC$. From the bearing, the line AC is at some angle. However, the problem implies A, B, C form a triangle where C is West of B. From the diagram, the angle at A between the two fences is $D\hat{A}E$.

$$\begin{aligned}\frac{BC}{\sin(D\hat{A}E)} &= \frac{AC}{\sin(A\hat{B}C)} \\ BC &= \frac{800 \cdot \sin(0.7195\,\mathrm{rad})}{\sin(105^\circ)} \\ &\approx \frac{800 \cdot 0.6589}{0.9659} \\ &\approx 545.72\,\mathrm{m}\end{aligned}$$

$\boxed{BC \approx 546\,\mathrm{m}}$

www.exam-mate.com

MathAA_21_SL_Summer_2021_Q9

Solution

Consider the function $f(x) = 90e^{-0.5x}$ for $x \in \mathbb{R}^+$. We are tasked with analyzing the geometric properties of its graph, its tangent line L, and the regions enclosed by these curves and the line $y = x$.

1. Finding the x-coordinate of P Point P is the intersection of $f(x)$ and the line $y = x$. We solve for x:

$$x = 90e^{-0.5x}$$

Using numerical methods to solve this transcendental equation:

$$\boxed{x \approx 6.56}$$

2. Exact coordinates of Q The line L is tangent to f at point Q and has a gradient of -1. We find the derivative of $f(x)$ using the **Chain Rule**:

$$\begin{aligned} f'(x) &= 90 \cdot (-0.5)e^{-0.5x} \\ &= -45e^{-0.5x} \end{aligned}$$

Setting the gradient to -1:

$$\begin{aligned} -45e^{-0.5x} &= -1 \\ e^{-0.5x} &= \frac{1}{45} \\ -0.5x &= \ln\left(\frac{1}{45}\right) = -\ln 45 \\ x &= 2\ln 45 \end{aligned}$$

To find the y-coordinate:

$$\begin{aligned} y &= f(2\ln 45) \\ &= 90e^{-0.5(2\ln 45)} \\ &= 90e^{-\ln 45} = 90 \cdot \frac{1}{45} = 2 \end{aligned}$$

The exact coordinates of Q are $\boxed{(2\ln 45, 2)}$.

3. Equation of line L Using the point-slope form $y - y_1 = m(x - x_1)$ with $m = -1$ and point $Q(2\ln 45, 2)$:

$$\begin{aligned} y - 2 &= -1(x - 2\ln 45) \\ y &= -x + 2\ln 45 + 2 \end{aligned}$$

This matches the required form.

4. Area of region A (i) Intersection of L and $y = x$:

www.exam-mate.com

$$x = -x + 2\ln 45 + 2$$
$$2x = 2\ln 45 + 2$$
$$x = \ln 45 + 1 \approx 4.8066$$

The x-coordinate of the intersection is $\boxed{\ln 45 + 1}$.

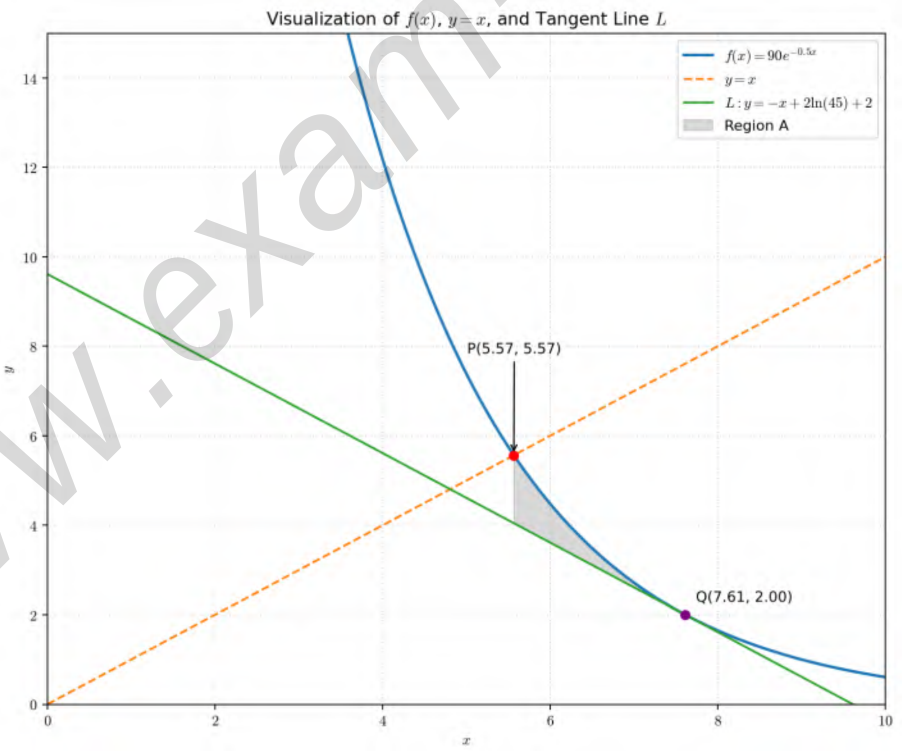
(ii) The area A is bounded by $y = x$, L, and $f(x)$. From the graph, the region is split by the intersection point $x_I = \ln 45 + 1$.

- From x_I to $x_P \approx 6.56$, the area is $\int_{x_I}^{x_P} (x - f(x))\,dx$.
- From x_I to $x_Q = 2\ln 45 \approx 7.61$, the area is $\int_{x_I}^{x_Q} (f(x) - L(x))\,dx$. Summing these components or using the geometry of the triangle formed by the lines:

$$\text{Area} = \int_{\ln 45+1}^{6.56} (x - 90e^{-0.5x})\,dx + \int_{6.56}^{2\ln 45} ((-x + 2\ln 45 + 2) - 90e^{-0.5x})\,dx$$
$$\approx \boxed{1.52}$$

5. Area enclosed by f, f⁻¹, and L The function f and its **inverse function** f^{-1} are symmetric about the line $y = x$. The line L has a gradient of -1, making it perpendicular to $y = x$. Due to this symmetry, the area enclosed by f, f^{-1}, and L is exactly twice the area A calculated in the previous step.

$$\text{Total Area} = 2 \times \text{Area } A$$
$$\approx 2 \times 1.5218$$
$$\approx 3.04$$



$\boxed{3.04}$

www.exam-mate.com